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A New Proof of the Jung–Abhyankar Theorem

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The Jung–Abhyankar theorem of the title asserts that if $P(Z) \in K[[X_1,..., X_n]][Z]$ is a Weierstrass polynomial over an algebraically closed field K of characteristic 0 with discriminant $D_Z(P) = X_1^{\alpha_1} \cdots X_r^{\alpha_r} \cdot \xi$, where $\xi(0,..., 0) \neq 0$, then the roots of $P(Z)$ are series of $[[X_1^{1/q},..., X_r^{1/q}, X_{r+1},..., X_n]]$ for some $q \in N - \{0\}$.

The first proof of this theorem, for the case $n = 2$, is due to Jung [3] and it uses topological methods, that is, $K = \mathbb{C}$ and $P(Z)$ and the roots are convergent series. Abhyankar [1] gave an algebraic proof for any n, based on some properties of the Galois group of the polynomial $P(Z)$ when its discriminant $D_Z(P)$ defines a divisor with normal crossing. If $n = 1$, then the theorem can be deduced from the classical Newton–Puiseaux theorem, *which has an elementary proof*, in the sense that it uses only some properties of the Newton polygon of P, (N.P.(P)) and the Weierstrass Preparation Theorem.

The key point of this work is the observation that if $P(Z)$ satisfies the hypothesis of the theorem, then its Newton polyhedron N.P.(P) has a property (cf. Definition 2 and Theorem 1, below) that allows us to extend the elementary proof of the Newton–Puiseaux theorem to the general case of the Jung–Abhyankar theorem.

Some consequences can be deduced from the proof. The first one is that in the case $K = \mathbb{C}$, $n > 2$ and the coefficients of $P(Z)$ are convergent series, then the roots are convergent too. The second consequence concerns the case when K is a non-algebraically closed field of characteristic 0. Then for each polynomial $P(Z)$ as above, there is a finite extension K' of K such that the roots of $P(Z)$ belong to $K'[[X_1^{1/q},..., X_r^{1/q}, X_{r+1},..., X_n]]$ for some $q \in N - \{0\}$.

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1. Quasiordinary and ν-Quasiordinary Weierstrass Polynomials

Let K be an algebraically closed field of characteristic 0. Let $P(Z) \in K[[X_1,..., X_n]][Z]$ be a Weierstrass polynomial (W-polynomial) and let $D_Z(P) \in K[[X_1,..., X_n]]$ be its discriminant with respect to Z.

Definition 1. The W-polynomial $P(Z)$ is *quasiordinary* (Q.O.) *with respect to* Z if

$$D_Z(P) = X_1^{\alpha_1} \cdots X_r^{\alpha_r} \cdot \xi(X_1,..., X_n)$$

with $\xi(0,..., 0) \neq 0$.

Given a set $A \subset N^{n+1}$, $|A|$ will denote the *convex hull* of $\bigcup_{\mathbf{a} \in A} (\mathbf{a} + N^{n+1})$.

We also write

$$H(P) = \{(i_1,..., i_{n+1}) \in N^{n+1} : P_{i_1,...,i_{n+1}} \neq 0\}$$

and N.P.$(P) = |H(P)|$, the Newton polyhedron of $P(Z)$. Let d be the degree of $P(Z)$ and $R_0 = (0,..., 0, d)$.

Definition 2. We say that $P(Z)$ is *ν-quasiordinary* (ν-Q.O.) *with respect to* Z if there is a point $R_1 \in$ N.P.(P), $R_1 \neq R_0$, such that if R_1' is the projection of R_1 over $N^n \times \{0\}$ from R_0 and $S = [R_0, R_1']$ is the segment *joining* R_0 *to* R_1', then

(1) N.P.$(P) \subset |S| = |[R_0, R_1']|$, and

(2) $P_S = \sum_{(i_1,...,i_{n+1}) \in S} P_{i_1 \cdots i_{n+1}} X_1^{i_1} \cdots X_n^{i_n} \cdot Z^{i_{n+1}}$

is a polynomial not a power of a linear form.

This concept was introduced by Hironaka [2, p. 45]. As we will see later (Theorem 1) every Q.O. W-polynomial $P(Z)$ having no coefficient in Z^{d-1} is ν-Q.O. The converse is not true as can be seen from the following example:

The polynomial

$$P(Z) = Z^4 - 2X_1X_2^2Z^2 + X_1^4X_2^4 + X_1^2X_2^7$$

is ν-Q.O. but not Q.O. since $X_1X_2(X_1^2 + X_2^3)$ is a factor of $D_Z(P)$.

Now let us consider the particular case where $P(Z)$ is a polynomial of $K[X_1,..., X_n][Z]$.

Proposition 1. *Let* $P(Z) = Z^d + A_2Z^{d-2} + \cdots + A_n \in K[X_1,..., X_n, Z]$ *be a Q.O. polynomial such that its coefficient in* Z^{d-1} *is* 0. *If* $D_Z(P) = u \cdot X_1^{a_1} \cdots X_n^{a_n}$, $u \in K^* = K - \{0\}$ *and* $R_1 = (a_1/(d-1),..., a_n/(d-1), 0)$, *then* $H(P)$ *is included into the segment* S *joining* R_0 *to* R_1.

Proof. First, assume $n=1$. Let P_1 be an irreducible factor of $P(Z)$ in $K[X_1, Z]$. $D_Z(P_1)$ divides to $D_Z(P)=u \cdot X_1^{a_1}$ and so $D_Z(P_1)=v \cdot X_1^{b_1}$, $b_1 \leqslant a_1$. Suppose $b_1 \neq 0$ and let $C_1 \subset A_2$ be the affine curve with equation $P_1(X_1, Z)=0$. The morphism $\pi\colon C_1 \to A_1$ induced by the first projection has just one ramification point, namely, the origin O of A_1, since the ramification points of π are exactly the zeros of the discriminant $D_Z(P_1)$.

If $p\colon \tilde{C}_1 \to C_1$ is a desingularization of C_1, then $\tilde{\pi}=\pi \cdot p$ is a morphism from $\tilde{C}_1$ into $P_1=A_1-\{O_\infty\}$ of degree d_1 (=degree (P_1)) with ramification locus $\operatorname{Ram}(\tilde{\pi}) \subset \{O, O_\infty\}$.

As the degree of $\tilde{\pi}$ is $d_1>1$, it follows from Riemann's genus formula that $g(\tilde{C}_1)=0$, that is, $\tilde{C}_1 \simeq P_1$, and $\operatorname{Ram}(\tilde{\pi})=\{O, O_\infty\}$. Moreover, we have $\#\tilde{\pi}^{-1}(O_\infty)=1$.

Let $\tilde{\pi}^{-1}(O_\infty)=Q$. Then $\tilde{C}_1-\{Q\}$ is isomorphic to A_1. Let $h\colon A_1 \to \tilde{C}_1-\{Q\}$ be such an isomorphism. Now $\tilde{\pi} \cdot h\colon A_1 \to A_1$ is a morphism of degree d_1 with one ramification point. Hence, it is clear that for an adequate affine coordinate T, $\tilde{\pi} \cdot h(T)=T^d$, thus $p \cdot h\colon A_1 \to C_1$ will be $p \cdot h(T)=(T^{d_1}, R(T))$ with $R(T) \in K[T]$ since $\pi(p \cdot h(T))=\tilde{\pi} \cdot h(T)=T^{d_1}$. So $P_1(T^{d_1}, R(T))=0$ and $\xi=R(X^{1/d_1})$ is a root of $P_1(X_1, Z)$. The fact that in the desingularization of C_1, $\#p^{-1}(O)=1$, implies that O is a unibranched singular point, and as a consequence, the roots of $P_1(X_1, Z)$ are conjugated of ξ. If $\xi=\xi_1, \xi_2, \ldots, \xi_{d_1}$ are all the roots of $P_1(X_1, Z)$, $\xi_i-\xi_j$ divides to $X_1^{b_1}$ and so $\xi_i-\xi_j=u_{ij}X^{r_{ij}/d_1}$. From the properties of the characteristic exponents it follows that all r_{ij}'s are equal and

$$\xi_i=u\eta_i \cdot X^{c_1/d_1}+H(X_1)$$

with $\eta_i^{d_1}=1$ and $H(X_1) \in K[X_1]$. If P_2 is another irreducible factor of P of degree d_2, in the same way one proves that

$$\bar{\xi}_j=\bar{\eta}_j \cdot u_2 X_1^{c_2/d_2}+G(X_1)$$

with $\bar{\eta}_j^{d_2}=1$ and $G(X_1) \in K[X_1]$. Now we have that

$$\xi_i-\bar{\xi}_j=\eta_1 u_1 X_1^{c_1/d_1}-\bar{\eta}_j u_2 X_1^{c_2/d_2}+H(X_1)-G(X_1)$$

is a divisor of $X_1^{a_1}$ and this is only possible if $c_1/d_1=c_2/d_2$ and $H(X_1)=G(X_1)$. But as $\#p^{-1}(O)=1$, $P_1(X_1, Z)$ is also irreducible in $K[[X_1]][Z]$, $\gcd(c_1, d_1)=1$, and so $c_1=d_1$ and $c_2=d_2$.

Hence we have that the roots of P are of the form

$$\xi_i=u_i X^{c_1/d_1}+H,$$

$$P_1(X_1, Z)=\prod_{i=1}^{s}(u_i X^{c_1/d_1}+H-Z).$$

Then it is clear that the coefficient in P of Z^{d-1} is $d \cdot H$ and by the hypothesis is O, i.e., $H = O$, and

$$P(X_1, Z) = \prod_{i-1}^{d} (u_i X^{c_1/d_1}),$$

that is, N.P.(P) is included into the segment joining $(0, d)$ to $(d \cdot (c_1/d_1), 0)$, but as $\xi_i - \xi_j = (u_i - u_j) X^{c_1/d_1}$ and in $D_Z(P) = X_1^{a_1}$ there are $d(d-1)$ factors $\xi_i - \xi_j$, $a_1 = d(d-1)(c_1/d_1)$ and $dc_1/d_1 = a_1/(d-1)$ as we wanted to prove.

If at the beginning of the proof $c_1 = 0$, i.e., $D_Z(P_1) = u$, the same reasoning shows that $d_1 = 1$ and $P(X_1, Z)$ belongs to $K[Z]$. So $D_Z(P) = u'$ and $a_1 = 0$. The result is the same since if $P(Z) \in K[Z]$, N.P.(P) is included into the segment $S = [(0, d), (0, 0)]$.

Now let $P(Z) \in K[X_1, ..., X_n, Z]$ with

$$D_Z(P) = u \cdot X_1^{a_1} \cdots X_n^{a_n}, \qquad u \in K^*,$$

and

$$S = \left[(0, ..., 0, d), \left(\frac{a_1}{d-1}, ..., \frac{a_n}{d-1}, 0\right)\right].$$

if $H(P) \neq S$, then there is a point $(i_1, ..., i_{n+1}) \in H(P)$ whose projection over $a_{n+1} = 0$ from $R_0 = (0, ..., 0, d)$ is

$$\left(\frac{d - i_{n+1}}{d} i_1, ..., \frac{d - i_{n+1}}{d} i_n, 0\right) \neq \left(\frac{a_1}{d-1}, ..., \frac{a_n}{d-1}, 0\right).$$

Assume $a_1/(d-1) \neq ((d - i_{n+1})/d)\, i_1$. We can look at $P(X_1, ..., X_n, Z)$ as a polynomial in X_1, Z with coefficients in the algebraic closure of $K(X_2, ..., X_n)$. As the coefficient in $X_1^{i_1} \cdot Z^{i_{n+1}}$ of P is not null, from the result for $n = 1$, we have that the projection of (i_1, i_{n+1}) from $(0, d)$ is $(a_1/(d-1), 0)$, which is a contradiction. ∎

Observe that in the above situation $P(Z)$ is ν-Q.O. because P is not a power of a linear form since $P(Z) \neq Z^d$. Now let us consider the general case, $P(Z)$ being a Weierstrass polynomial.

THEOREM 1. *Let $P(Z) \in K[[X_1, ..., X_n]][Z]$ be a Q.O. W-polynomial such that its coefficient in Z^{d-1} is 0. Then $P(Z)$ is ν-Q.O. with respect to Z.*

Proof. In the initial hypothesis, if there is an $R_1 \in$ N.P.(P) such that N.P.$(P) \subset |[R_0, R_1']|$, then $P(Z)$ is ν-Q.O. since condition (2) of the definition *holds automatically* as $P(Z)$ has no term in Z^{d-1}.

Let $\pi_0 \colon N^{n+1} \to N^n$ be the projection from R_0 over the hyperplane

$a_{n+1}=0$. If $P(Z)$ is not ν-Q.O. then there is no $R_1 \in$ N.P.(P) such that N.P.$(P) \subset |[R_0, R_1']|$ and this is equivalent to π_0(N.P.(P)) being a convex set not of the form $\bar{R}+N^n$, i.e., π_0(N.P.(P)) has at least two vertices.

We claim the existence of a linear form $L=a_1x_1+\cdots+a_nx_n-k$, $a_i \in Q$, $a_i>0$, $i=1,\ldots,n$, and $k>0$ such that

$$L(\pi_0(\text{N.P.}(P))) \geqslant 0$$

and

$$L(\bar{R}_1)=L(\bar{R}_2)=0$$

for some $\bar{R}_1$, $\bar{R}_2 \in \pi_0$(N.P.(P)).

Let $L_k=x_1+\cdots+x_n-k$. As $\pi_0(\text{N.P.}(P)) \subset N^n$ and $(0,\ldots,0) \notin \pi_0(\text{N.P.}(P))$, there exists a $k \in N$ such that

$$L_k(\pi_0(\text{N.P.}(P))) \geqslant 0$$

and

$$L_k(\bar{R}_1)=0 \qquad \text{for some} \quad \bar{R}_1 \in \pi_0(\text{N.P.}(P)),$$

if there is an $\bar{R}_2 \neq \bar{R}_1$ such that

$$L_k(\bar{R}_2)=0, \qquad \text{take} \quad L=L_k.$$

Otherwise, as π_0(N.P.(P)) has at least two vertices, $\bar{R}_1+N^n \neq \pi_0$(N.P.(P)). If $\bar{R}_1=(r_1,\ldots,r_n)$, there is an $\bar{R}_2=(r_1',\ldots,r_n') \in \pi_0$(N.P.$(P)$) such that for some i, $r_i'<r_i$. Assume $r_1'<r_1$ and consider

$$H_\lambda=(x_1-r_1)+\lambda(x_2+\cdots+x_n-r_2-\cdots-r_n).$$

If $\lambda=1$, $H_1=L_k$, and if

$$\lambda_0=\frac{r_2'+\cdots+r_n'-r_2-\cdots-r_n}{r_1-r_1'}$$

(notice that $\lambda_0>0$ since $r_1-r_1'>0$ and $r_2'+\cdots+r_n'>k-r_1=r_2+\cdots+r_n$) then

$$H_\lambda(\bar{R}_2)=H_\lambda(\bar{R}_1)=0.$$

Since $\lambda_0<1$, the set $(N^n \cap (L_0(\bar{R}) \geqslant 0)) \cap (L_{\lambda_0}(\bar{R}) \geqslant 0)$ is compact and therefore there is a $\lambda_1 \geqslant \lambda_0$ such that $L_{\lambda_1}(\pi_0(\text{N.P.}(P))) \geqslant 0$ and there is an $\bar{R}' \neq \bar{R}_1$ with $L_{\lambda_1}(\bar{R}')=0$. Thus L_{λ_1} would be the desired form.

Let $H=a_1x_1+\cdots+a_{n+1}x_{n+1}-c=0$ be the equation of the hyperplane of N^{n+1} containing $R_0=(0,\ldots,0,d)$ and including the hyperplane $L=0$

of N^n. We may assume $a_i \in N$. It is clear that for each $1 \leqslant i \leqslant n-1$, $a_i > 0$ and if R_1 and R_2 are the points projecting on $\bar{R}_1$ and $\bar{R}_2$,

$$H(R_0) = H(R_1) = H(R_2) = 0$$

and

$$H(\text{N.P.}(P)) \geqslant 0.$$

In this situation, we will show the existence of a curve with parametric equation

$$x_1 = h_1(u), \dots, x_n = h_n(u)$$

with $h_i(u) \neq 0$ such that $D_Z(P)(h_1(u), \dots, h_n(u)) = 0$. To this end, consider

$$P_H = \sum_{(i_1, \dots, i_{n+1}) \in H} P_{i_1, \dots, i_{n+1}} X_1^{i_1} \cdots X_n^{i_n} \cdot Z^{i_{n+1}}.$$

P_H is a polynomial since $\{H = O\} \subset N^{n+1}$ is a finite set because $a_i > 0$.

Let $D_Z(P_H)$ be the discriminant of P_H. If $D_Z(P_H) \neq 0$, there exists a factor $G(X_1, \dots, X_n)$ of $D_Z(P_H)$ different from X_i, $i = 1, \dots, n$, since the points R_0, R_1 and R_2 are not collinear (Proposition 1). Thus there is a point $(c_1, \dots, c_n) \in K^n$ such that $c_i \neq 0$ for each $i = 1, \dots, n$, and

$$D_Z(P_H)(c_1, \dots, c_n) = 0.$$

If $D_Z(P_H) = 0$, this is trivially true.

Now assume $P_H \neq P$. It follows from the definition of $D_Z(P_H)$ that there is $c_{n+1} \in K$ such that

$$P_H(c_1, \dots, c_{n+1}) = 0 \qquad \text{and} \qquad \frac{\delta P_H}{\delta Z}(c_1, \dots, c_{n+1}) = 0. \tag{*}$$

Changing variables by

$$\begin{aligned} X_1 &= (c_1 + \bar{X}_1) \cdot T^{a_1} \\ &\vdots \\ X_n &= (c_n + \bar{X}_n) \cdot T^{a_n} \\ Z &= (c_{n+1} + \bar{Z}) \cdot T^{a_{n+1}} \end{aligned}$$

we get

$$\begin{aligned} &P((c_1 + \bar{X}_1)\, T^{a_1}, \dots, (c_{n+1} + \bar{Z})\, T^{a_{n+1}}) \\ &\quad = T^k \cdot (P_1((c_1 + \bar{X}_1), \dots, (c_{n+1} + \bar{Z})) + TH(\bar{X}_1, \dots, \bar{Z})) \\ &\quad = T^k Q(\bar{X}_1, \dots, \bar{X}_n, \bar{Z}, T) \end{aligned}$$

since the monomial of $P - P_H$ has weight with respect to H greater than k. Moreover, $Q(0,..., 0) = 0$. Similarly,

$$\frac{\delta P}{\delta Z}((c_1 + \bar{X}_1)\, T^{a_1},..., (c_{n+1} + \bar{Z})\, T^{a_{n+1}})$$

$$= T^{k-a_{n+1}} \cdot \left(\frac{\delta P_H}{\delta Z}((c_1 + \bar{X}_1),...) + TH\right)$$

$$= T^{k-a_{n+1}} \cdot Q'(X_1,..., T)$$

with $Q'(0,..., 0) = 0$.

Let V be the algebroid variety of K^{n+2} defined by the equations $Q = Q' = 0$. Then V is not included into the hypersurface W with equation $T = 0$, since if $V \subset W$, $V = V \cap W$ and from (1) and (2), the equations of $V \cap W$ are

$$T = 0,$$

$$P_H((c_1 + \bar{X}_1),..., (c_{n+1} + \bar{Z})) = 0,$$

$$\frac{\delta P_H}{\delta P}((c_1 + \bar{X}_1),..., (c_{n+1} + \bar{Z})) = 0.$$

Clearly, we can choose an $(n + 1)$-tuple $(c_1,..., c_{n+1})$ such that on the one hand one component of dimension $n-1$ of $\{P_H = (\delta P_H/\delta Z) = 0\}$ passes through the point $(c_1,...,c_{n+1})$ and, on the other hand, every component of V has dimension $\geqslant n$, but this is a contradiction. So $V \not\subset W$ and there is an algebroid curve $C \subset V$, $C \not\subset W$. If $\bar{X}_1 = g_1(u),..., T = g_{n+2}(u)$ is a parametrization of T, $g_i(0) = 0$ and $g_{n+2}(u) \neq 0$, and from (1)–(2) writing $h_i(u) = (c_i + g_i(u))\, g_{n+2}(u)^{a_i}$, $i = 1,..., n + 1$,

$$P(h_1,..., h_{n+1}) = 0$$

and

$$\frac{\delta P}{\delta Z}(h_1,..., h_{n+1}) = 0.$$

Keeping in mind the definition of the discriminant (see [4]), we have

$$D_Z(P)(h_1,..., h_n) = 0$$

and $h_i(u) \neq 0$, $1 \leqslant i \leqslant n$. If $P_H = P$, then $h_i = c_i u^{a_i}$ is such that $D_Z(P)(h_1,..., h_n) = 0$, and in both cases we get a contradiction since

$$D_Z(P) = X_1^{a_1} \cdots X_n^{a_n} \cdot \xi(X_1,..., X_n)$$

with

$$\xi(0,\dots,0) \neq 0$$

as

$$D_Z(P)(h_1,\dots,h_n) = h_1^{a_1} \cdots h_n^{a_n} \cdot \xi(h_1,\dots,h_n) \neq 0.$$

Therefore $P(Z)$ is ν-Q.O. ∎

Remark 1. If $P(Z)$ is ν-Q.O., N.P.$(P) \subset \|[R_0, R_1']\|$, and the coordinates of R_1' are $(b_1,\dots,b_n,0)$, we say that $P(Z)$ is ν-Q.O. *with exponent* $(b_1,\dots,b_n)$. As $D_Z(P)$ is a weighted polynomial in the coefficients of $P(Z)$, it follows that $D_Z(P)$ has $X_1^{b_1(d-1)} \cdots X_n^{b_n(d-1)}$ as a factor. In particular, if $P(Z)$ is also Q.O. and $D_Z(P) = X_1^{a_1} \cdots X_n^{a_n}$, then $a_i \geqslant b_i(d-1)$. Therefore $b_i \neq 0$ implies $a_i \neq 0$.

The converse is not true: for example,

$$P(Z) = Z^4 - 2X_1^3(1+X_2)\,Z^2 + X_1^6(1-X_2)^2$$

is a ν-Q.O. W-polynomial with exponent $(6,0)$ and its discriminant is

$$D_Z(P) = X_1^{18} \cdot X_2^2 \cdot \xi(X_1, X_2) \qquad \text{with} \quad \xi(0,0) \neq 0.$$

2. The Jung–Abhyankar Theorem

Notice that if $n=1$, then every Q.O. W-polynomial is automatically ν-Q.O. This fact together with the Weierstrass Preparation Theorem (W.P.T.) are the fundamental points in the proof of the Newton–Puiseaux Theorem. Now Theorem 1 allows us to give a proof of the Jung–Abhyankar Theorem valid for the general case $(n \geqslant 1)$ following the ideas of the proof of the Newton–Puiseaux Theorem.

Theorem 2 (Jung–Abhyankar's Theorem). *Let $P(Z) \in K[[X_1,\dots,X_n]][Z]$ be a W-polynomial such that*

$$D_Z(P) = X_1^{a_1} \cdots X_r^{a_r} \cdot \xi(X_1,\dots,X_n)$$

with $\xi(0,\dots,0) \neq 0$. Then the roots of $P(Z)$ belong to $K[[X_1^{1/q},\dots,X_r^{1/q}, X_{r+1},\dots,X_n]]$ for some $q \in N - \{0\}$.

Proof. If $P(Z) = Z^d + A_1 Z^{d-1} + \cdots + A_n$ with $A_i \in K[[X_1,\dots,X_n]]$, then we put $Z = Z' - (1/d)A_1$ and the coefficient in Z'^{d-1} of $P(Z')$ is null. This change does not modify the discriminant and so we can assume that $P(Z)$ fulfils the hypothesis of Theorem 1.

The proof will run by induction on d.

If $d=2$, then

$$(P(Z)=Z^2+X_1^{b_1}\cdots X_r^{b_r}\cdot\xi)$$

with $\xi(0,\dots,0)=V\neq 0$. Let u be a root of $P_S(1,\dots,1,Z)=Z^2+V$. If $Z=u+\bar{Z}$, then $(u+\bar{Z})^2+V=\bar{Z}^2+2u\bar{Z}$ and with the following change of variables

$$Z=(u+\bar{Z})\cdot X_1^{b_1/2}\cdots X_r^{b_r/2} \qquad (*)$$

we get

$$\begin{aligned}
&P'(X_1^{1/2},\dots,X_r^{1/2},X_{r+1},\dots,X_n,\bar{Z})\\
&\quad=P(X_1,\dots,X_n,(u+\bar{Z})\,X_1^{b_1/2}\cdots X_r^{b_r/2})\\
&\quad=X_1^{b_1}\cdots X_r^{b_r}\cdot(\bar{Z}^2+2u\bar{Z}(1+\cdots)+H(X_1^{1/2}\cdots))\\
&\quad=X_1^{b_1}\cdots X_r^{b_r}\cdot\bar{P}.
\end{aligned}$$

Clearly, $\bar{P}(0,\dots,0,\bar{Z})=\bar{Z}^2+2u\bar{Z}$. But there is a K-isomorphism from $K[[X_1^{1/2},\dots,X_r^{1/2},X_{r+1},\dots,X_n]]$ into $K[[X_1,\dots,X_n]]$ sending $X_i^{1/2}$ to X_i, $1\leqslant i\leqslant r$. This isomorphism preserves the factorization and we can apply the Implicit Function Theorem (I.F.T.) to $\bar{P}(\bar{Z})$ to guarantee the existence of a $\phi\in K[[X_1^{1/2},\dots,X_r^{1/2},X_{r+1},\dots,X_n]]$ such that

$$\bar{P}(X_1^{1/2},\dots,X_r^{1/2},X_{r+1},\dots,X_n,\phi)=0.$$

From $(*)$ it follows that

$$\xi=(u+\phi)\,X_1^{b_1/2}\cdots X_r^{b_r/2}$$

is a root of $P(Z)$.

Now assume that the degree of $P(Z)$, d, is greater than 2. By Theorem 1, $P(Z)$ is ν-Q.O. and by Remark 1 its exponent is of the form $(b_1,\dots,b_r,0,\dots,0)=R_1$. Let S be the segment $[R_0,R_1]$. Then N.P.$(P)\subset|S|$ and $P_S=Z^d+\cdots+v\cdot X_1^{b_1}\cdots X_r^{b_r}$ is not a power of a linear form.

Let u be a root of $P_S(1,\dots,1,Z)$. As $A_1=0$, u is a root with multiplicity $d_1<d$ and as in the above case if

$$Z=(u+\bar{Z})\,X_1^{b_1/d}\cdots X_1^{b_r/d} \qquad (**)$$

and P' is the polynomial resulting from this change of variables, it follows from N.P.$(P)\subset|S|$ that

$$P'(X_1^{1/d},\dots,X_r^{1/d},X_{r+1},\dots,X_n,\bar{Z})=X_1^{b_1}\cdots X_r^{b_r}\cdot\bar{P}. \qquad (***)$$

The coefficient in $\bar{Z}^{d_1}$ of the polynomial $\bar{P}$ is different from 0 and in fact we have

$$\bar{P}(0,..., 0, \bar{Z}) = \bar{Z}^d + \cdots + d\bar{Z}^{d_1}.$$

If $d_1 = 1$, we appeal to the I.F.T. as before and the proof ends similarly to the case $d = 2$.

If $d_1 > 1$, by the same reason as above, using the W. P. Theorem there are $\bar{P}_1$, $\bar{P}_2 \in K[[X_1^{1/d},..., X_r^{1/d}, X_{r+1},..., X_n]]$ such that $\bar{P} = \bar{P}_1 \cdot \bar{P}_2$. Moreover $\bar{P}_1$ is a W-polynomial of degree d_1.

Keeping in mind (**) and (***), it follows from the elementary properties of the discriminants that

$$X_1^{a_1} \cdots X_r^{a_r} \cdot \xi = D_Z(P) = X_1^{c_1/d} \cdots X_r^{c_r/d} \cdot D_{\bar{Z}}(\bar{P}),$$

whence

$$D_{\bar{Z}}(\bar{P}) = X_1^{e_1/d} \cdots X_r^{e_r/d} \cdot \bar{\xi}$$

with $\bar{\xi}(0,..., 0) \neq 0$, that is, $\bar{P}$ is Q.O. and as $D_Z(\bar{P}_1)$ is a factor $D_{\bar{Z}}(\bar{P})$, $\bar{P}_1$ is also Q.O. in $K[[X_1^{1/d},..., X_r^{1/d}, X_{r+1},..., X_n]]$.

If p is the K-isomorphism sending $X_i^{1/d}$ to X_i, then $p(\bar{P}_1)$ is a Q.O. W-polynomial of degree $d_1 < d$.

By the induction hypothesis, $p(\bar{P}_1)$ has a root $\psi \in K[[X_1^{1/m},..., X_r^{1/m}, X_{r+1},..., X_n]]$. Then

$$\bar{q} = p^{-1}(\psi) \in K[[X_1^{1/d \cdot m},..., X_r^{1/d \cdot m}, X_{r+1},..., X_n]]$$

is a root of $\bar{P}_1$, whence by (**) and (***)

$$\phi = (u + \bar{q})\, X_1^{b_1/d} \cdots X_r^{b_r/d}$$

is a root of $P(Z)$. The conjugated roots of ϕ, $\phi_2,..., \phi_s$, obtained by means of the changes $X_i \to \eta_i \cdot X_i$, $\eta_i^{m \cdot d} = 1$ are also roots of $P(Z)$ and $P_1(Z) = \prod_{i=1}^{s} (Z - \phi_i)$ is an irreducible W-polynomial. Now each factor of $P(Z)$ is Q.O., and following this process with another factor of $P(Z)$, $P(Z)$ can be split into linear factors in $K[[X_1^{1/q},..., X_r^{1/q}, X_{r+1},..., X_n]]$ for some $q \geqslant 1$. ∎

An immediate consequence of the above proof is that if $K = \mathbb{C}$ or a complete valued and algebraically closed field, then starting with $K\{\{X_1,..., X_n\}\}$, the ring of convergent series, we get convergent series, that is,

Corollary 1. *Let $P(Z) \in K\{\{X_1,..., X_n\}\}[Z]$ be a Q.O. W-polynomial. Then the roots of $P(Z)$ are members of $K\{\{X_1^{1/q},..., X_r^{1/q}, X_{r+1},..., X_n\}\}$ for some $q \geqslant 1$.*

Proof. It can be checked easily that if in the above proof we start with a convergent series, the resulting series are convergent too, and as we know that by using the I.F. Theorem and the W.P. Theorem we get convergent series, then the roots are convergent. ∎

Now let K be a non-algebraically closed field of characteristic 0.

PROPOSITION. *Let $P(Z) \in K[[X_1,..., X_n]][Z]$ be a Q.O. W-polynomial. Then there is a finite extension $K' \supset K$ such that the roots of $P(Z)$ are in $K'[[X_1^{1/q},..., X_r^{1/q}, X_{r+1},..., X_n]]$ for some $q \geqslant 1$.*

Proof. In the proof of Theorem 2, in order to continue the process it is enough to add a root of $P_S(1,..., 1, Z)$ to K. It is clear that since in each step degree $(\bar{P}) <$ degree(P), by adding finitely many roots to K. We get a root of $P(Z)$ in $K_1[[X_1^{1/q},..., X_r^{1/q}, X_{r+1},..., X_n]]$. If we add to K_1 every qth root of the unit, then we have an extension $K_2 \supset K_1$ where an irreducible factor of $P(Z)$ splits linearly. By making the same operation with the others, we get the result. ∎

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